

HKN ECE 310 Exam 1 Review

Corey Snyder and Mois Bourla

UIUC

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- LSIC Systems
- BIBO Stability
- Impulse Response and Convolution
- Z-transform



For our system T ,

- Linearity: must satisfy *homogeneity* and *additivity*.
 - Homogeneity: $T(ax) = aT(x)$
 - Additivity: $T(x + z) = T(x) + T(z)$
 - Can be summarized by *superposition*:
 $T(ax + bz) = aT(x) + bT(z)$
- Shift Invariance: shifting the input shifts the output by the same amount.
 - If $T(x[n]) = y[n]$, then $T(x[n - n_0]) = y[n - n_0]$.
 - $T(x[n - n_0])$ means we shift our input by n_0 : we replace every n inside our input arguments with $n - n_0$.
 - $y[n - n_0]$ means we shift our output by n_0 : we replace every *function* of n in our output with $f(n) - n_0$.
- Causality: output cannot depend on future input values.



Three ways to check for BIBO stability:

- 1 Absolute summability of the **impulse response**:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- 2 System definition:

- Given $|x[n]| < \alpha$, if $|T(x[n])| = |y[n]| < \beta < \infty$, T is BIBO stable
- In other words, a bounded input yields a bounded output.
- For example: $y[n] = x^5[n] + 3$ vs. $y[n] = x[n] * u[n]$

- 3 Pole-zero plot (more on this soon)



$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n - m] = \sum_{m=-\infty}^{\infty} h[m]x[n - m]$$

The following statements are equivalent:

- Our system T is LSI.
- Our system T can be described by its impulse response $h[n]$.
- Our system response to an input signal is a convolution.

Also,

- If x is of length L and h is of length M , y must be of length $L + M - 1$.



Impulse Response

Let $x[n]$ be the input to an LSI system with impulse response $h[n]$. Then the system output $y[n]$ is given by:

$$y[n] = x[n] * h[n]$$

By the identity property of convolution, we can find $h[n]$ by passing a Kronecker delta $\delta[n]$ to our system:

$$h[n] = \delta[n] * h[n]$$

For example:

$$y[n] = 2x[n] - 3x[n - 1] + x[n - 2]$$

$$h[n] = 2\delta[n] - 3\delta[n - 1] + \delta[n - 2]$$



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Typically perform inverse z-transform by inspection or by partial fraction decomposition
- Important properties:
 - Multiplication by n : $nx[n] \leftrightarrow -z \frac{dX(z)}{dz}$
 - Delay property: $y[n-k]u[n-k] \leftrightarrow z^{-k}Y(z)$
- Make sure to note the Region of Convergence (ROC) for your transforms, a Z-transform is not unique without one!
- Convolution theorem: $y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{Z}} X(z)H(z) = Y(z)$



- An LSI system is BIBO stable if its ROC contains the unit circle.
- For causal systems, the ROC is anything *greater than* the outermost pole: $|z| > |p_{\max}|$
- For a non-causal system, the ROC is anything *less than* the innermost pole: $|z| < |p_{\min}|$
- If we sum multiple systems the ROC is the *intersection* of each system's ROC.



But what if the ROC is $|z| > 1$ or $|z| < 1$?

- This system is *marginally stable*... though sometimes we just call the system unstable anyway.
- For unstable systems, you are commonly asked to find a bounded input that yields an unbounded output. Few ways to do this:
 - Pick an input that excites the poles of a marginally stable system. This means we pick an input with frequency equal to the angle of the pole. For example, pole at $z = j$ is excited by $e^{j\frac{\pi}{2}n}$ or $\cos\left(\frac{\pi}{2}n\right)$
 - If the system's impulse response $h[n]$ is not absolutely summable, $u[n]$ works.
 - If the system's impulse response is unbounded, $\delta[n]$ works.



For the following systems, determine whether is is linear, shift-invariant and causal.

- 1 $y[n] = |x[n]|$
- 2 $y[n] = nx[n]$
- 3 $y[n] = x[n - 1] + x[|n|]$
- 4 $y[n] = \log(x[n])$



LSIC Examples: Part 1

$y[n] = |x[n]|$ is non-linear, shift-invariant, and causal.

- Linearity:

$$T(ax) \stackrel{?}{=} aT(x)$$

$$T(ax) = |ax[n]| \neq a|x[n]| = aT(x) \\ \implies \text{Non-linear}$$

- Shift-invariance:

$$T(x[n - n_0]) \stackrel{?}{=} y[n - n_0]$$

$$T(x[n - n_0]) = |x[(n - n_0)]| = |x[(n) - n_0]| = y[n - n_0] \\ \implies \text{Shift-invariant}$$

- Causality: our output time sample is always equal to our input time sample; therefore, this system is causal.



LSIC Examples: Part 2

$y[n] = nx[n]$ is linear, shift-varying, and causal.

- Linearity:

$$T(ax + bz) \stackrel{?}{=} aT(x) + bT(z)$$

$$\begin{aligned} T(ax + bz) &= n(ax[n] + bz[n]) = anx[n] + bnz[n] = aT(x) + bT(z) \\ &\implies \text{Linear} \end{aligned}$$

- Shift-invariance:

$$T(x[n - n_0]) \stackrel{?}{=} y[n - n_0]$$

$$\begin{aligned} T(x[n - n_0]) &= nx[(n - n_0)] \neq (n - n_0)x[(n) - n_0] = y[n - n_0] \\ &\implies \text{Shift-varying} \end{aligned}$$

- Causality: our output time sample is always equal to our input time sample; therefore, this system is causal. Don't for the n out front does not affect causality since it is not part of the input signal's argument!



$y[n] = x[n - 1] + x[|n|]$ is linear, shift-varying, and non-causal.

- Linearity:

$$T(ax + bz) \stackrel{?}{=} aT(x) + bT(z)$$

$$\begin{aligned} T(ax + bz) &= (ax[n - 1] + bz[n - 1]) + (ax[|n|] + bz[|n|]) \\ &= ax[n - 1] + ax[|n|] + bz[n - 1] + bz[|n|] = aT(x) + bT(z) \\ &\implies \text{Linear} \end{aligned}$$

- Shift-invariance:

$$T(x[n - n_0]) \stackrel{?}{=} y[n - n_0]$$

$$\begin{aligned} T(x[n - n_0]) &= x[(n - n_0) - 1] + x[|n - n_0|] \\ &\neq x[(n - 1) - n_0] + x[|(n) - n_0|] = y[n - n_0] \\ &\implies \text{Shift-varying} \end{aligned}$$

- Non-causal: for example, $y[-3]$ relies on $x[3]$ in the future.



$y[n] = \log(x[n])$ is non-linear, shift-invariant, and causal.

- Linearity:

$$\begin{aligned} T(ax) &\stackrel{?}{=} aT(x) \\ T(ax) = \log(ax[n]) &\neq a \log(x[n]) = aT(x) \\ &\implies \text{Non-linear} \end{aligned}$$

- Shift-invariance:

$$\begin{aligned} T(x[n - n_0]) &\stackrel{?}{=} y[n - n_0] \\ T(x[n - n_0]) = \log(x[(n - n_0)]) &= \log(x[(n) - n_0]) = y[n - n_0] \\ &\implies \text{Shift-invariant} \end{aligned}$$

- Causality: our output time sample is always equal to our input time sample; therefore, this system is causal.



BIBO Stability Example

For each of the following systems defined either by an input-output relationship or impulse response, determine whether the system is BIBO stable or not:

① $h[n] = \delta[n]$

② $h[n] = \left(-\frac{1}{3}\right)^n u[n]$

③ $y[n] = x^2[n] + 1$

④ $y[n] = (x[n])^a + b, 0 < a, b < c < \infty$



BIBO Stability Example

- ① $h[n] = \delta[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.



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BIBO Stability Example

- 1 $h[n] = \delta[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
- 2 $h[n] = \left(-\frac{1}{3}\right)^n u[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
- 3 $y[n] = x^2[n] + 1$: If $x[n] < c \implies x^2[n] + 1 < c^2 + 1 < \infty$; therefore, BIBO stable.



BIBO Stability Example

- 1 $h[n] = \delta[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
- 2 $h[n] = \left(-\frac{1}{3}\right)^n u[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
- 3 $y[n] = x^2[n] + 1$: If $x[n] < c \implies x^2[n] + 1 < c^2 + 1 < \infty$; therefore, BIBO stable.
- 4 $y[n] = (x[n])^a + b$, $0 < a, b < c < \infty$:
 $x[n] < c \implies (x[n])^a < c^a$. Moreover, adding $b < c < \infty$ will keep our output bounded below $c^a + b < \infty$; therefore, BIBO stable.



Impulse Response and Convolution Examples

Given $x[n] = [1 \ 2 \ 3 \ 2 \ 9 \ 8 \ 9]$ and $h[n] = [-1 \ 0 \ 1]$, compute $y[n] = x[n] * h[n]$. Bonus: what does this filter do?

Suppose we have a digital filter $h[n]$ with an unknown impulse response. We do know the system output to the following two input signals. Determine the impulse response in terms of the two system outputs.

- $x_1[n] = [2 \ 4 \ 2 \ 4] \rightarrow y_1[n]$
- $x_2[n] = [0 \ 2 \ 1 \ 2] \rightarrow y_2[n]$



Impulse Response and Convolution Examples

Part 1: $x[n] = [1 \ 2 \ 3 \ 2 \ 9 \ 8 \ 9]$ and $h[n] = [-1 \ 0 \ 1]$

$$y[n] = x[n] * h[n] = [-1 \ -2 \ -2 \ 0 \ -6 \ -6 \ 0 \ 8 \ 9]$$



Impulse Response and Convolution Examples

Part 1: $x[n] = [1 \ 2 \ 3 \ 2 \ 9 \ 8 \ 9]$ and $h[n] = [-1 \ 0 \ 1]$

$$y[n] = x[n] * h[n] = [-1 \ -2 \ -2 \ 0 \ -6 \ -6 \ 0 \ 8 \ 9]$$

Part 2:

- $x_1[n] = [2 \ 4 \ 2 \ 4] \rightarrow y_1[n]$

- $x_2[n] = [0 \ 2 \ 1 \ 2] \rightarrow y_2[n]$

$$\begin{aligned}h[n] &= h[n] * \delta[n] \\ &= h[n] * \left(\frac{1}{2}x_1[n] - x_2[n]\right) \\ &= h[n] * \frac{1}{2}x_1[n] - h[n] * x_2[n] \\ &= \frac{1}{2}y_1[n] - y_2[n].\end{aligned}$$



Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- 1 Suppose we pass input $x[n]$ to system $h[n]$ which is the connection of $h_1[n]$ and $h_2[n]$ in series. Write the output $y[n]$ in terms of $x[n]$, $h_1[n]$ and $h_2[n]$.
- 2 Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of $h[n]$?
- 3 Suppose the two systems are now connected in parallel to form $h[n]$. Now what is the resulting transfer function and impulse response of $h[n]$?



Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose we pass input $x[n]$ to system $h[n]$ is the connection of $h_1[n]$ and $h_2[n]$ in series. Write the output $y[n]$ in terms of $x[n]$, $h_1[n]$ and $h_2[n]$.

$$y[n] = x[n] * h_1[n] * h_2[n]$$



Combinations of Systems: Part 2

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of $h[n]$?

$$h[n] = h_1[n] * h_2[n]$$

$$\downarrow \mathcal{Z}$$

$$H(z) = H_1(z)H_2(z)$$

$$H_1(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$$

$$H_2(z) = \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}}$$



Notice that $h_2[n] = \left(\frac{1}{3}\right)^n u[n-1] = \frac{1}{3} \left(\frac{1}{3}\right)^{n-1} u[n-1]$

Combinations of Systems: Part 2

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of $h[n]$?

$$\begin{aligned} H(z) &= \frac{\frac{1}{3}z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)} \\ &= \frac{K_1}{1 - \frac{1}{2}z^{-1}} + \frac{K_2}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

$$\xrightarrow{\text{PFD}} K_1 = 2, \quad K_2 = -2$$

Inspection/Tables



$$h[n] = 2 \left(\frac{1}{2}\right)^n u[n] - 2 \left(\frac{1}{3}\right)^n u[n]$$



Combinations of Systems: Part 3

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n], \quad h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose the two systems are now connected in parallel to form $h[n]$. Now what is the resulting transfer function and impulse response of $h[n]$?

$$\begin{aligned} h[n] &= h_1[n] + h_2[n] \\ &= \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n-1] \end{aligned}$$

$$\begin{aligned} H(z) &= H_1(z) + H_2(z) \\ &= \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$



Marginal Stability Example

Suppose we have a system response given by $H(z) = \frac{1}{1+z^{-3}}$. Which of the following bounded inputs would cause this system to have an unbounded output? There may be more than one!

- ① $\cos\left(\frac{2\pi}{3}n\right)u[n]$
- ② $\cos\left(\frac{\pi}{3}n\right)u[n]$
- ③ $u[n]$
- ④ $e^{-j\frac{\pi}{3}n}u[n]$
- ⑤ $(-1)^n u[n]$



Marginal Stability Example

$$H(z) = \frac{1}{1 + z^{-3}}$$

First, must find the poles of $H(z)$:

$$1 + z^{-3} = 0$$

$$z^{-3} = -1 = e^{-j\pi} = e^{-j(\pi+2\pi k)}, \quad \forall k \in \mathbb{Z}$$

$$z = e^{j\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)}$$

$$z_1 = e^{j\frac{\pi}{3}}, \quad z_2 = e^{j\pi}, \quad z_3 = e^{j\frac{5\pi}{3}}$$

Now let's look at our signals...



Marginal Stability Example

$$H(z) = \frac{1}{1 + z^{-3}}$$

$$z_1 = e^{j\frac{\pi}{3}}, \quad z_2 = e^{j\pi}, \quad z_3 = e^{j\frac{5\pi}{3}}$$

- ❶ $\cos\left(\frac{2\pi}{3}n\right) u[n]$
- ❷ $\cos\left(\frac{\pi}{3}n\right) u[n]$ ✓
- ❸ $u[n]$
- ❹ $e^{-j\frac{\pi}{3}n} u[n]$ ✓
- ❺ $(-1)^n u[n]$ ✓



Thank you

Good luck studying and good luck on your exam!

