HKN ECE 310 Exam 2 Review Session

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Topics

- DTFT and Frequency Response
- Ideal Sampling and Reconstruction
- DFT and FFT

Discrete Time Fourier Transform

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_d(\omega)e^{j\omega n}d\omega$$

- Important Properties:
  - **Periodicity by 2π!**
  - Linearity
  - Symmetries (Magnitude, angle, real part, imaginary part)
  - Time shift and modulation
  - Product of signals and convolution
  - Parseval's Relation

Frequency Response

- For any **stable** LSI system: $H_d(\omega) = H(z)|_{z=e^{j\omega}}$
- What is the physical interpretation of this?
  - The DTFT is the z-transform evaluated along the unit circle!
- Why is the frequency response nice to use in addition to the z-transform?
  - $e^{j\omega}$ is an *eigenfunction* of LSI systems
    - $h[n] * Ae^{j\omega_0} = \lambda Ae^{j\omega_0} = AH_d(\omega_0)e^{j\omega_0}$
  - By extension:
    $x[n] = \cos(\omega_0 n + \theta) \rightarrow y[n] = |H_d(\omega_0)| \cos(\omega_0 n + \theta + \angle H_d(\omega_0))$

Magnitude and Phase Response

- Very similar to ECE 210
- Frequency response, and all DTFTs for that matter, are 2π periodic
- Magnitude response is fairly straightforward
  - Take the magnitude of the frequency response, remembering that $|e^{j\omega}| = 1$
- For phase response:
  - Phase is "contained" in $e^{j\omega}$ terms
  - Remember that cosine and sine introduce sign changes in the phase
    - When a cosine or sine changes phase, we have a contribution of $\pm\pi$ phase.
  - Limit your domain from $-\pi$ to π.
- For **real-valued** signals and systems:
  - Magnitude response is even-symmetric
  - Phase response is odd-symmetric

DTFT Exercise 1

- Let our signal be

$$h[n] = \{\mathbf{1}, 2, 1\}.$$

a) Compute the DTFT of $h[n]$.

b) Plot the magnitude response.

c) Plot the phase response.

DTFT Exercise 2

- Suppose we have a new system defined by a real-valued impulse response $h[n]$ with corresponding DTFT $H_d(\omega)$. We also know the following about the magnitude and phase responses:

$$|H_d(\omega)| = \begin{cases} 1, & \pi \le \omega < \frac{3\pi}{2} \\ 2, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases} \qquad \angle H_d(\omega) = \begin{cases} -\frac{\pi}{4}, & \pi \le \omega < \frac{3\pi}{2} \\ \frac{\pi}{4}, & \frac{3\pi}{2} \le \omega < 2\pi \end{cases}$$

a) Plot the magnitude response of $H_d(\omega)$ on the interval $-\pi$ to π.

b) Plot the phase response of $H_d(\omega)$ on the interval $-\pi$ to π.

Sinusoidal Response Exercise 1

- We have an LSI system defined by the following LCCDE:

$$y[n] = x[n] - 2x[n-1] + x[n-2].$$

a) Find $H(z)$.

b) Find $H_d(\omega)$.

c) Find the output $y[n]$ to each of the following inputs:

   i. $x_1[n] = 2 + \cos(\pi n)$

   ii. $x_2[n] = e^{j\frac{\pi}{4}n} + \sin\left(-\frac{\pi}{2}n\right)$

Ideal A/D Conversion

- Sampling via an impulse train will yield infinitely many copies of the analog spectrum in the digital frequency domain

$$X_d(\omega) = \frac{1}{T}\sum_{k\in\mathbb{Z}} X_a\left(\frac{\omega - 2\pi k}{T}\right)$$

- Important relations to recall:
- Nyquist Sampling Theorem:

$$\frac{1}{T} = f_s > 2B = 2f_{max}$$

- Relationship between digital ω and analog frequencies Ω:

$$\omega = \Omega T$$

Ideal D/A Conversion

- Recall that our DTFT has infinitely many copies of our sampled analog spectrum.

- Ideal D/A conversion requires we perfectly recover only the central copy between $-\pi$ and π.
  - Digital signal is given notion of continuous-time back with a sampling period T.
  - We suppose that we have an ideal low-pass analog filter ("interpolation filter") with cutoff frequency corresponding to $\frac{\pi}{T}$.

Sampling Exercise 1

- Suppose we sampled some analog signal defined by

$$x_a(t) = \cos(\Omega_0 t)$$

with sampling period $T = \frac{1}{1000}$ s to obtain the digital signal $x[n] = \cos\left(\frac{\pi}{4}n\right)$. Which of the following are possible values for Ω_0? (There may be more than one!)

a) 250π rad/s

b) $\frac{\pi}{4000}$ rad/s

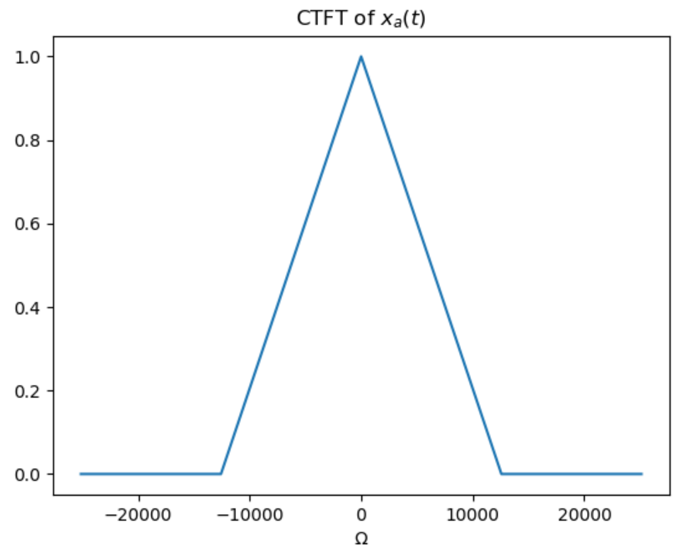
c) -1750π rad/s

d) 4250π rad/s

e) $\frac{1}{8}$ rad/s

Sampling Exercise 2

- We have an analog signal $x_a(t)$ with CTFT $X_a(\Omega)$ with maximum frequency 4000π.



For each of the following sampling periods T, draw the sampled DTFT spectrum $X_d(\omega)$ on the interval -3π to 3π.

a) $T_1 = \frac{1}{8000}\ s$

b) $T_2 = \frac{1}{4000}\ s$

c) $T_3 = \frac{1}{2000}\ s$

Discrete Fourier Transform

- $$X[k] = \sum_{n=0}^{N-1} x[n] e^{\frac{-j2\pi kn}{N}}, 0 \le k \le N-1$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\frac{j2\pi kn}{N}}, 0 \le n \le N-1$$

- What is the relationship between the DTFT and the DFT?

$$\omega_k = \frac{2\pi k}{N}$$

$$\omega_k \in \left\{0, \frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2\pi(N-1)}{N}\right\}$$

DFT Properties

- Periodicity by N
- *Circular* shift
- *Circular* modulation
- *Circular* convolution
- We must amend our DTFT properties with the “circular” term because the DFT is defined over a finite length signal and assumes periodic extension of that finite signal.

Zero-Padding

- We can improve the **resolution** of the DFT simply by adding zeros to the end of the signal.
- This doesn't change the frequency content of the DTFT!
  - No information/energy is being added.
- Instead, it increases the number of samples the DFT takes of the DTFT.
- This can be used to improve spectral resolution.

Windowing

- Recall that the DFT implies infinite periodic extension of our signal.
- This extension can lead to artifacts known as "spectral leakage"
- Window functions help with these artifacts
  - Rectangular window
  - Hamming window
  - Hanning window
  - Kaiser window
- Windowing is just multiplication in the time domain

$$x_w = x[n]w[n]$$

- We care about the main lobe width and side lobe attenuation of these windows.
  - In particular, know the tradeoffs between the rectangular and Hamming windows

Fast Linear Convolution via FFT

- Convolution in the time domain requires $O(n^2)$ operations.
- By convolution theorem, perhaps we can do better in the frequency domain?
- Don't forget multiplication in DFT domain is *circular* convolution in time.
- To avoid aliasing, we adopt the following procedure
- Given signal x and filter h of lengths N and L, respectively:

1. Zero-pad x and h to length $N + L - 1$
2. Take their FFTs
3. Multiply in frequency domain
4. Take the inverse FFT

This procedure takes $O(n \log n)$ operations.

DFT Exercise 1

- Surprisingly, we have another signal

$$x[n] = \cos\left(\frac{\pi}{3}n\right), 0 \leq n < 18.$$

a) For which value(s) of k is the DFT of $x[n]$, $X[k]$, largest?

b) Suppose now that we zero-pad our sequence with 72 zeros to obtain $y[n]$. For which value(s) of k is $Y[k]$ largest?