HKN ECE 313 EXAM 2 REVIEW SESSION

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EXAM 2 TOPICS

- Continuous-type Random Variables (mean and variance of CRVs)
- Uniform Distribution
- Exponential Distribution
- Poisson Process
- Linear Scaling of PDFs
- Gaussian Distribution
- ML Parameter Estimation for Continuous Random Variables
- Functions of a random variable
- Failure Rate Functions
- Binary Hypothesis Testing
- Joint CDFs, PMFs, and PDFs
- Independence of Random Variables
- Distributions of sums of random variables

CONTINUOUS-TYPE RANDOM VARIABLES

- Cumulative Distribution Functions (CDFs)
  - Must be non-decreasing
  - $F_X(-\infty) = 0, F_X(+\infty) = 1$
  - Must be right continuous
- $F_X(c) = \int_{-\infty}^{c} f_x(u)du$
- $P\{X = a\} = 0$
- $P\{a < X \leq b\} = F_X(b) - F_X(a) = \int_{a}^{b} f_X(u)du$
- $E[X] = \mu_x = \int_{-\infty}^{+\infty} u f_x(u)du$

UNIFORM DISTRIBUTION

- $Unif(a, b)$: All values between a and b are equally likely
- pdf: $f(u) = \begin{cases} \frac{1}{b-a} & a \leq u \leq b \\ 0 & else \end{cases}$
- mean: $\frac{a+b}{2}$
- variance: $\frac{(b-a)^2}{12}$

EXPONENTIAL DISTRIBUTION

- Exponential(λ): limit of scaled geometric random variables
- pdf: $f(t) = \lambda e^{-\lambda t} \; t \geq 0$
- mean: $\frac{1}{\lambda}$
- variance: $\frac{1}{\lambda^2}$
- Memoryless Property
  - $P\{T \geq s + t \mid T \geq s\} = P\{T \geq t\} \; s, t \geq 0$

POISSON PROCESS

- Poisson process is used to model the number of counts in a time interval. Similar to how the exponential distribution is a limit of the geometric distribution, the Poisson process is the limit of the Bernoulli process.
  - Bonus: A Poisson process is a collection of i.i.d exponential random variables.
- If we have a rate λ and time duration t, the number of counts $\mathrm{N_t} \sim Pois(\lambda t)$
- pdf: $f_{N_t}(k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$
- mean: λt
- variance: λt
- Furthermore, $N_t - N_s \sim Pois\big(\lambda(t-s)\big), (t > s)$,
- Disjoint intervals, e.g. $s = [0,2]$ and $t = [2,3]$, are independent. This property is both important and remarkable. In fact, we shape our analysis of Poisson processes around this frequently. (More on this later!)

LINEAR SCALING OF PDFS

- $If\ Y = aX + b$:

- $E[Y] = aE[X] + b$

- $Var(Y) = \ a^2 Var(X)$

- $f_y(v) = \ f_x(\frac{v-b}{a})\frac{1}{a}$

GAUSSIAN DISTRIBUTION

- Gaussian (or Normal) Distribution $\sim N(\mu, \sigma^2)$
  - Standard Gaussian, $\hat{X}: \mu = 0, \sigma = 1$
- $pdf: f(u) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}$
- $mean: \mu$
- $variance: \sigma^2$
- If we standardize our Gaussian, where: $\hat{X} = \frac{X-\mu}{\sigma}$
  - $\Phi(c) = \int_{-\infty}^{c} f(u)du$
  - $Q(c) = 1 - \Phi(c) = \int_{c}^{\infty} f(u)du$
  - $\Phi(-c) = Q(c)$

ML PARAMETER ESTIMATION

- Suppose we have a random variable with a given distribution/pdf that depends on a parameter, θ. By taking trials of the random variable, we can estimate θ by finding the value that maximizes the likelihood of the observed event, $\hat{\theta}_{\text{ML}}$.
- There are a few ways we can find $\hat{\theta}_{\text{ML}}$
  - Take derivative of provided pdf and set it equal to zero (maximization)
  - Observe the intervals where the likelihood increases and decreases, and find the maximum between these intervals
  - Intuition!

FUNCTIONS OF RANDOM VARIABLES

- Suppose $Y = g(X)$, and we want to be able to describe the distribution of Y
- Step 1: Identify the support of X. Sketch the pdf of X and g. Identify the support of Y. Determine whether Y is a Continuous or Discrete RV
  - Take a deep breath! You've done some important work here.
- Step 2 (for CRV): Use the definition of the CDF to find the CDF of Y:
  - $F_Y(c) = P\{Y \leq c\} = P\{g(X) \leq c\}$
- Step 2 (for DRV): Find the pmf of Y directly using the definition of the pmf
  - $p_Y(v) = P\{Y = v\} = P\{g(X) = v\}$
- Step 3 (for CRV): Differentiate the CDF of Y in order to find the pdf of Y

GENERATING A RV WITH A SPECIFIED DISTRIBUTION

- We can generate any distribution by applying a function to a uniform distribution
- This function should be the inverse of the CDF of the desired distribution
- Ex: if we want an exponential distribution,
  - $F_X(c) = 1 - e^{-c} = u$; then find $F_X^{-1}(c)$

FAILURE RATE FUNCTIONS

- We can assess the probability of a failure in a system through a failure rate function, $h(t)$.
- $F(t) = 1 - e^{-\int_0^t h(s)ds}$
- Two popular failure rate functions:
  - Consistent lifetime
  - "Bath tub"

BINARY HYPOTHESIS TESTING

- Similar to BHT with Discrete RVs
- Maximum Likelihood (ML) Rule
  - $\Lambda(\mathrm{k}) = \frac{f_1(k)}{f_0(k)}$
  - $\Lambda(k) = \begin{cases} > 1 \; declare \; H_1 \; is \; true \\ < 1 \; declare \; H_0 \; is \; true \end{cases}$
- Maximum a Posteriori (MAP) Rule
  - Prior probabilities: $\pi_0 = P(H_0), \pi_1 = P(H_1)$
  - $H_1 true \; if \; \pi_1 f_1(k) > \pi_0 f_0(k)$, same as $\Lambda(\mathrm{k}) = \frac{f_1(k)}{f_0(k)} > \tau \; where \; \tau = \frac{\pi_0}{\pi_1}$
- Probabilities of False Alarm and Miss
  - $p_{false\ alarm} = P(Say\ H_1 \mid H_0\ is\ true)$
  - $p_{miss} = P(Say\ H_0 \mid H_1\ is\ true)$
  - $p_e = \pi_1 p_{miss} + \pi_o p_{false\ alarm}$

JOINT CDF, PMF, AND PDF

DISCRETE RANDOM VARIABLES

- Joint CDF
  - $F_{X,Y}(u_o, v_o) = P\{X \le u_o, Y \le v_o\}$
- Joint PMF
  - $p_{X,Y}(u_o, v_o) = P\{X = u_o, Y = v_o\}$
- Marginal PMFs
  - $p_X(u) = \sum_j p_{X,Y}(u, v_j)$
  - $p_Y(v) = \sum_i p_{X,Y}(u_i, v)$
- Conditional PMFs
  - $p_{(Y|X)}(v|u_o) = P(Y = v|X = u_o) = \frac{p_{X,Y}(u_o,v)}{p_X(u_o)}$

CONTINUOUS RANDOM VARIABLES

- Joint CDF
  - $F_{X,Y}(u_o, v_o) = P\{X \le u_o, Y \le v_o\}$
- Joint PDF
  - $F_{X,Y}(u_o, v_o) = \iint f_{X,Y}(u, v)dvdu$
- Marginal PDFs
  - $f_X(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v)dv$
  - $f_Y(v) = \int_{-\infty}^{+\infty} f_{X,Y}(u, v)du$
- Conditional PDFs
  - $f_{(Y|X)}(v|u_o) = \frac{f_{X,Y}(u_o,v)}{f_X(u_o)}$

INDEPENDENCE OF JOINT DISTRIBUTIONS

- We can check independence in a joint distribution in a couple ways:

- $f_{X,Y}(u,v) = f_X(u)f_Y(v)$

- The support of $f_{X,Y}(u,v)$ is a *product set*
  - Product set must have the swap property, which is satisfied if:
    - $(a,b)\ and\ (c,d) \in support\left(f_{X,Y}\right),\ and\ then\ (a,d)\ and\ (c,b)\ are\ also \in support(f_{X,Y})$
  - Checking for a product set is only sufficient to prove *dependence*. Saying that the support of the joint pdf is a product set is not sufficient to check independence

DISTRIBUTION OF SUMS OF RANDOM VARIABLES

- Suppose we want to find the distribution from the sum of two independent random variables where $Z = X + Y$

- The pdf or pmf of Z is the *convolution* of the two pdfs/pmfs

- So…

WHAT IS CONVOLUTION?

- A beautiful, extraordinary linear operator that describes natural phenomena in a fundamental and concise manner.

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- A beautiful, extraordinary linear operator that describes natural phenomena in a fundamental and concise manner. But for real, given $Z = X + Y$
- Discrete (pmfs):

$$p_z(u) = p_x(u) * p_y(u)$$

$$p_z(u) = \sum_{k=-\infty}^{\infty} p_x(k) p_y(u-k) = \sum_{k=-\infty}^{\infty} p_x(u-k)\, p_y(k)$$

- Continuous (pdfs):

$$f_z(u) = f_x(u) * f_y(u)$$

$$f_z(u) = \int_{-\infty}^{\infty} f_x(\tau) f_y(u-\tau) d\tau = \int_{-\infty}^{\infty} f_x(u-\tau) f_y(\tau) d\tau$$

FA15 PROBLEM 2

- Let N_t be a Poisson process with rate $\lambda > 0$
- a. Obtain $P\{N_3 = 5\}$
- b. Obtain $P\{N_7 - N_4 = 5\}$ and $E[N_7 - N_4]$
- c. Obtain $P\{N_7 - N_4 = 5 | N_6 - N_4 = 2\}$
- d. Obtain $P\{N_6 - N_4 = 2 | N_7 - N_4 = 5\}$

FA14 PROBLEM 3

3. **[12 points]** Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ.

   (a) (6 points) Express $E[N(t)N(t+s)]$, $s, t > 0$ as a function of λ, s, and t.

   (b) (6 points) Let $\lambda = 2$ arrivals/hour and assume that the Poisson process models the arrival of customers into a post office. Find the probability of the following event, which involves three conditions:
   (Three customers arrive between 1 and 3pm,
   one customer arrives between 2 and 3pm,
   and one customer arrives between 2 and 4pm.)

FA15 PROBLEM 4

- Let the joint pdf for the pair (X, Y) be:

$$f_{X,Y}(x,y) = \begin{cases} cxy, 0 \leq x \leq 1, 0 \leq y \leq 1, x + y \leq 1 \\ 0, \qquad\qquad\qquad\qquad\qquad otherwise \end{cases}$$

- a. Compute the marginal $f_X(x)$. You can leave it in terms of c.
- b. Obtain the value of the constant c for $f_{X,Y}$ to be a valid joint pdf.
- c. Obtain $P\{X + Y < \frac{1}{2}\}$
- d. Are X and Y independent? Explain why or why not.

SU16 PROBLEM 3

- Let X be Gaussian with mean -1 and variance 16
- a. Express $P\{X^3 \leq -8\}$ in terms of the Φ function
- b. Let $Y = \frac{1}{2}X + \frac{3}{2}$. Sketch the pdf of Y, $f_y(v)$, clearly marking important points
- c. Express $P\{Y \geq \frac{1}{2}\}$ in terms of the Φ function

SP15 PROBLEM 4

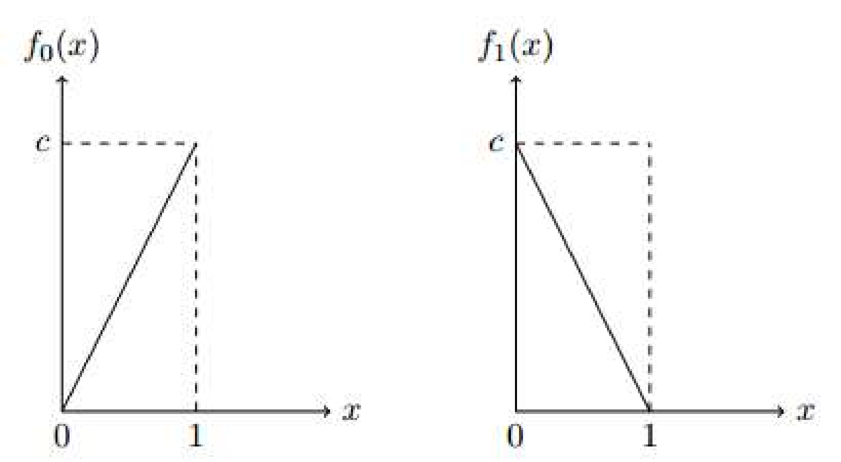
4. [**16 points**] The two parts of this problem are unrelated.

   (a) Let X be uniformly distributed in $[0, 1]$. Find the CDF for

$$Y = 2|X - 1/2|.$$

FA15 PROBLEM 3

- Let X be a continuous-type random variable taking values in $[0,1]$. Under hypothesis H_0 the pdf of X is f_0. Under hypothesis H_1 the pdf of X is f_1. Both pdfs are plotted below. The priors are known to be $\pi_0 = 0.6$ and $\pi_1 = 0.4$.



- a. Find the value of c
- b. Specify the maximum a posteriori (MAP) decision rule for testing H_0 vs. H_1.
- c. Find the error probabilities $p_{false\ alarm}$, $p_{miss\ detection}$, and the average probability of error p_e for the MAP rule.

FA12 PROBLEM 6

6. **[24 points]** Suppose random variables X and Y have the joint probability density function (pdf): $f_{XY}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{elsewhere} \end{cases}$

(a) (4 points) Are X and Y independent? Explain your answer.

(b) (6 points) Determine the marginal pdf of X, $f_X(u)$.

(c) (3 points) For what values of u is the conditional pdf of Y given $X = u$, $f_{Y|X}(v|u)$, well defined?

(d) (4 points) Determine $f_{Y|X}(v|u)$ for the values of u for which it is well defined. Be sure to indicate where its value is zero.

(e) (7 points) Determine $P\{Y > X\}$.