HKN ECE 313 Exam 2 Review Solutions

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FA15 Problem 2

FA15 Problem 2: Part (a)

Obtain $P(N_3 = 5)$:

$$N_3 \sim \text{Pois}(3\lambda)$$

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$$N_3 \sim \text{Pois}(3\lambda)$$
$$P(N_3 = 3) = \frac{e^{-3\lambda}(3\lambda)^5}{5!}$$

FA15 Problem 2: Part (b)

Obtain $P(N_7 - N_4 = 3)$ and $\mathbb{E}[N_7 - N_4]$:

$$N_7 - N_4 \sim \ \text{Pois}(3\lambda)$$

FA15 Problem 2: Part (b)

Obtain $P(N_7 - N_4 = 3)$ and $\mathbb{E}[N_7 - N_4]$:

$$N_7 - N_4 \sim \text{ Pois}(3\lambda)$$

$$P(N_7 - N_4 = 5) = P(N_3 = 5) = \frac{e^{-3\lambda}(3\lambda)^5}{5!}$$

$$\mathbb{E}[N_7 - N_4] = 3\lambda \text{ (defn. of Poisson random variable)}$$

FA15 Problem 2: Part (c)

Obtain $P(N_7 - N_4 = 5 | N_6 - N_4 = 2)$:

$$P(N_7 - N_4 = 5 | N_6 - N_4 = 2) = \frac{P(N_7 - N_4 = 5 \cap N_6 - N_4 = 2)}{P(N_6 - N_4 = 2)}$$

FA15 Problem 2: Part (c)

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\begin{aligned}
P(N_7 - N_4 = 5 | N_6 - N_4 = 2) &= \frac{P(N_7 - N_4 = 5 \cap N_6 - N_4 = 2)}{P(N_6 - N_4 = 2)} \\
&= \frac{P(N_7 - N_6 = 3 \cap N_6 - N_4 = 2)}{P(N_6 - N_4 = 2)}
\end{aligned}
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&= \frac{P(N_7 - N_6 = 3) P(N_6 - N_4 = 2)}{P(N_6 - N_4 = 2)}
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FA15 Problem 2: Part (d)

Obtain $P(N_6 - N_4 | N_7 - N_4 = 5)$:

$$P(N_6 - N_4 = 2 | N_7 - N_4 = 5)$$
$$= \frac{P(N_7 - N_4 = 5 | N_6 - N_4 = 2) P(N_6 - N_4 = 2)}{P(N_7 - N_4 = 5)}$$

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&= \frac{P(N_1 = 3) P(N_2 = 2)}{P(N_3 = 5)} \\
&= \frac{\left(\frac{e^{-\lambda}\lambda^3}{3!}\right)\left(\frac{e^{-2\lambda}(2\lambda)^2}{2!}\right)}{\frac{e^{-3\lambda}(3\lambda)^5}{5!}} \\
&= \frac{40}{243}
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Does not depend on λ!

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$$= \frac{P(N_1 = 3) P(N_2 = 2)}{P(N_3 = 5)}$$

$$= \frac{\left(\frac{e^{-\lambda}\lambda^3}{3!}\right)\left(\frac{e^{-2\lambda}(2\lambda)^2}{2!}\right)}{\frac{e^{-3\lambda}(3\lambda)^5}{5!}}$$

$$= \frac{40}{243}$$

Does not depend on λ! Wow!

FA14 Problem 3

FA14 Problem 3: Part (a)

Express $\mathbb{E}[N_t N_{t+s}], s, t > 0$ as a function of λ, s, and t:

This is a tricky question that requires a common math trick: adding a clever form of zero and regrouping.

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$$\begin{aligned}\mathbb{E}[N_t N_{t+s}] &= \mathbb{E}[N_t(N_{t+s} - N_t + N_t)] \\ &= \mathbb{E}[N_t(N_{t+s} - N_t) + N_t^2]\end{aligned}$$

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Notice that N_t and $N_{t+s} - N_t$ count over disjoint intervals since the latter counts from time t to time $t + s$! Thus,

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Notice that N_t and $N_{t+s} - N_t$ count over disjoint intervals since the latter counts from time t to time $t+s$! Thus,

$$\begin{aligned}\mathbb{E}[N_t N_{t+s}] &= \mathbb{E}[N_t]\mathbb{E}[N_{t+s} - N_t] + \mathbb{E}[N_t^2] \\ &= (\lambda t)(\lambda s) + \mathrm{Var}(N_t) + \mathbb{E}^2[N_t] \\ &= \lambda^2 st + \lambda t + \lambda^2 t^2\end{aligned}$$

FA14 Problem 3: Part (b)

Let $\lambda = 2$ arrivals/hour and assume that customer arrivals are Poisson. Let A be the event three customers arrive from 1-3pm; B, one customer from 2-3pm; and C, one customer from 2-4pm.

Compute $P(ABC)$:

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- Thus, an equivalent set of events would be:
  - $A' =$ two customers arrive between 1 and 2pm.
  - $B' = B =$ one customer arrives between 2 and 3pm.
  - $C' =$ no customers arrives between 3 and 4pm.

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- These new events are now disjoint but equivalent s.t. $P(ABC) = P(A'B'C') = P(A')P(B')P(C')$.

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$$P(ABC) = \left(\frac{e^{-2}2^2}{2!}\right)\left(\frac{e^{-2}2^1}{1!}\right)\left(\frac{e^{0}2^0}{0!}\right)$$
$$= 4e^{-6}.$$

FA15 Problem 4

FA15 Problem 4: Part (a)

Given $f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1 \\ 0, & \text{else} \end{cases}$,

Compute the marginal $f_X(x)$:

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Compute the marginal $f_X(x)$:

The support of (X, Y) is a triangle with vertices $(0,0), (1,0), (0,1)$.

$$
\begin{aligned}
f_X(x) &= \int_0^{1-x} cxy\,dy \\
&= \frac{c}{2}(xy^2)\Big|_0^{1-x} \\
&= \frac{c}{2}x(1-x)^2
\end{aligned}
$$

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Given $f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1 \\ 0, & \text{else} \end{cases}$,

Compute c s.t. $f_{X,Y}$ is a valid pdf:

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Also works to make $f_X(x)$ a valid marginal pdf.

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Compute c s.t. $f_{X,Y}$ is a valid pdf:

Also works to make $f_X(x)$ a valid marginal pdf.

$$\begin{aligned} \int_0^1 f_X(x) &= \int_0^1 \frac{c}{2}(x^3 - 2x^2 + x)dx \\ &= \frac{c}{2}\left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 \\ &= \frac{c}{24} \\ &= 1 \\ \implies c = 24. \end{aligned}$$

FA15 Problem 4: Part (c)

Given $f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0, & \text{else} \end{cases}$,

Obtain $P\left(X+Y < \frac{1}{2}\right)$:

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Given $f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1 \\ 0, & \text{else} \end{cases}$,

Obtain $P\left(X+Y<\frac{1}{2}\right)$:

$$P\left(X+Y<\frac{1}{2}\right) = \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} 24xy\,dy\,dx$$

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$$= \int_0^{\frac{1}{2}} 12x^3 - 12x^2 + 3x\,dx$$

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$$\begin{aligned} P\left(X+Y<\frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}-x} 24xy dy dx \\ &= \int_0^{\frac{1}{2}} 12x^3 - 12x^2 + 3x dx \\ &= \frac{1}{16}. \end{aligned}$$

FA15 Problem 4: Part (d)

Given $f_{X,Y}(x,y) = \begin{cases} cxy, & 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ x+y \leq 1 \\ 0, & \text{else} \end{cases}$,

Are X and Y independent? Why or why not?

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Are X and Y independent? Why or why not?

X and Y are not independent since $\frac{f_{X,Y}}{f_X}$ is clearly not solely a function of Y. Thus, we cannot have it that $f_{X,Y} = f_X f_Y$ and X and Y cannot be independent.

SU16 Problem 3

Let $X \sim \mathcal{N}(-1, 16)$.

Express $P(X^3 \leq -8)$ in terms of the Φ function:

SU16 Problem 3: Part (a)

Let $X \sim \mathcal{N}(-1, 16)$.

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$$P(X^3 \leq -8) = P(X \leq -2)$$

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Let $X \sim \mathcal{N}(-1, 16)$.

Express $P(X^3 \leq -8)$ in terms of the Φ function:

$$\begin{aligned} P(X^3 \leq -8) &= P(X \leq -2) \\ &= P\left(\frac{X+1}{4} \leq \frac{-2+1}{4}\right) \end{aligned}$$

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$$\begin{aligned} P(X^3 \leq -8) &= P(X \leq -2) \\ &= P\left(\frac{X+1}{4} \leq \frac{-2+1}{4}\right) \\ &= P\left(\hat{X} \leq -\frac{1}{4}\right) \\ &= \Phi\left(-\frac{1}{4}\right) \end{aligned}$$

SU16 Problem 3: Part (b)

Let $X \sim \mathcal{N}(-1, 16)$.

Let $Y = \frac{1}{2}X + \frac{3}{2}$. Sketch $f_Y(v)$, mark important points:

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- Recall that a Gaussian is fully explained by its μ and σ^2.

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- The key piece here is to **mark important points**.
- Recall that a Gaussian is fully explained by its μ and σ^2.
- μ locates the mean.
- σ^2 determines the height of the Gaussian
  - height $= \frac{1}{\sqrt{2\pi\sigma^2}}$.

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- μ locates the mean.
- σ^2 determines the height of the Gaussian
  - height $= \frac{1}{\sqrt{2\pi\sigma^2}}$.
- By our linear transformation formulas: $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$
  - $\mu_Y = \frac{1}{2}\mu_X + \frac{3}{2} = 1$
  - $\sigma_Y^2 = \left(\frac{1}{2}\right)^2 \sigma_X^2 = 4$

SU16 Problem 3: Part (b)

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  - $\mu_Y = \frac{1}{2}\mu_X + \frac{3}{2} = 1$
  - $\sigma_Y^2 = \left(\frac{1}{2}\right)^2 \sigma_X^2 = 4$
- Draw a Gaussian centered at μ_Y, indicate the height is $\frac{1}{\sqrt{2\pi\sigma_Y^2}}$ and you're all good!

SU16 Problem 3: Part (c)

Let $X \sim \mathcal{N}(-1, 16)$.

Express $P\left(Y \geq \frac{1}{2}\right)$ in terms of the Φ function:

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$$P\left(Y \geq \frac{1}{2}\right) = P\left(\frac{Y-1}{2} \geq \frac{\frac{1}{2}-1}{2}\right)$$
$$= P\left(\hat{Y} \geq -\frac{1}{4}\right)$$

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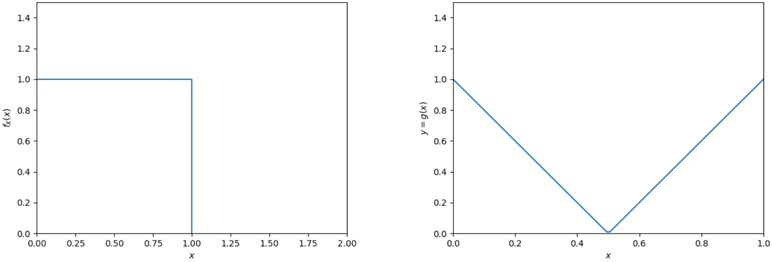
&= P\left(\hat{Y} \geq -\frac{1}{4}\right) \\
&= Q\left(-\frac{1}{4}\right) \\
&= \Phi\left(\frac{1}{4}\right)
\end{aligned}
$$

SP15 Problem 4

SP15 Problem 4: Step 1

Identify support of X and Y, sketch $f_X(x)$ and $g(Y)$

- $X \in [0, 1]$
- $Y \in [0, 1]$
- Y is continuous.



(a) $f_X(x)$ (b) $g(x)$

Figure 1: PDF of X and $Y = g(X)$.

SP15 Problem 4: Step 1(b)

Take a deep breath!

SP15 Problem 4: Step 2

Use defn. of CDF to find $F_Y(c)$:

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$$F_Y(c) = P(Y \leq c)$$

SP15 Problem 4: Step 2

Use defn. of CDF to find $F_Y(c)$:

$$\begin{aligned} F_Y(c) &= P(Y \leq c) \\ &= P(2|X - \frac{1}{2}| \leq c) \\ &= P\left(\frac{-c+1}{2} \leq X \leq \frac{c+1}{2}\right) \end{aligned}$$

SP15 Problem 4: Step 2

Use defn. of CDF to find $F_Y(c)$:

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\begin{aligned}
F_Y(c) &= P(Y \leq c) \\
&= P(2|X - \frac{1}{2}| \leq c) \\
&= P\left(\frac{-c+1}{2} \leq X \leq \frac{c+1}{2}\right) \\
&= \int_{\frac{-c+1}{2}}^{\frac{c+1}{2}} f_X(x)dx \\
&= \begin{cases} 0, & c < 0 \\ c, & 0 \leq c < 1 \\ 1, & c \geq 1 \end{cases}
\end{aligned}
$$

SP15 Problem 4: Step 3

Differentiate the CDF of Y to find the PDF:

$$f_Y(c) = \frac{dF_Y(c)}{dc}$$
$$= \begin{cases} 1, & 0 \leq c \leq 1 \\ 0, & \text{else} \end{cases}$$

$$Y \sim \text{Uni}(0, 1)!$$

FA15 Problem 3

FA15 Problem 3: Part (a)

Let $X \in [0,1]$ be a CRV. Under H_0, $X \sim f_0 = cx$ and under $X \sim f_1 = c(1-x)$. We have $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

Find the value of c.

$$\begin{aligned} \int_0^1 f_0(x)dx &= \int_0^1 cx dx \\ &= \frac{1}{2}cx^2\Big|_0^1 \\ &= \frac{1}{2}c \\ &= 1 \end{aligned}$$

$$\implies c = 2$$

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Specify the MAP decision rule for H_0 vs. H_1.

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$$\Lambda(k) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

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Specify the MAP decision rule for H_0 vs. H_1.

$$\Lambda(k) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$$

$$\frac{f_1(k)}{f_0(k)} \gtrless \frac{\pi_0}{\pi_1}$$

FA15 Problem 3: Part (b)

Let $X \in [0,1]$ be a CRV. Under H_0, $X \sim f_0 = cx$ and under $X \sim f_1 = c(1-x)$. We have $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

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$$\frac{f_1(k)}{f_0(k)} \gtrless \frac{\pi_0}{\pi_1}$$

$$\frac{2-2k}{2k} \gtrless \frac{3}{2}$$

$$k \underset{H_1}{\overset{H_0}{\gtrless}} \frac{2}{5}$$

FA15 Problem 3: Part (c)

Let $X \in [0, 1]$ be a CRV. Under H_0, $X \sim f_0 = cx$ and under $X \sim f_1 = c(1 - x)$. We have $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

Find error probabilities $p_{\text{fa}}, p_{\text{miss}}$ and p_e for the MAP rule:

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$$p_{\text{fa}} = P(\text{Say } H_1 | H_0 \text{ true}) = P_{H_0}\left(X \leq \frac{2}{5}\right)$$

$$= \int_0^{\frac{2}{5}} 2x dx = \frac{4}{25}$$

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$$p_{\text{miss}} = P(\text{Say } H_0 | H_1 \text{ true}) = P_{H_1}\left(X > \frac{2}{5}\right)$$

$$= \int_{\frac{2}{5}}^{1} 2 - 2x dx = \frac{9}{25}$$

FA15 Problem 3: Part (c)

Let $X \in [0,1]$ be a CRV. Under H_0, $X \sim f_0 = cx$ and under $X \sim f_1 = c(1-x)$. We have $\pi_0 = 0.6$ and $\pi_1 = 0.4$.

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$$= \int_0^{\frac{2}{5}} 2x dx = \frac{4}{25}$$

$$p_{\text{miss}} = P(\text{Say } H_0 | H_1 \text{ true}) = P_{H_1}\left(X > \frac{2}{5}\right)$$

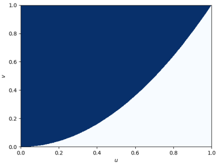
$$= \int_{\frac{2}{5}}^{1} 2 - 2x dx = \frac{9}{25}$$

$$p_e = \pi_0 p_{\text{fa}} + \pi_1 p_{\text{miss}} = \frac{6}{25}$$

FA12 Problem 6

FA12 Problem 6: Part (a)

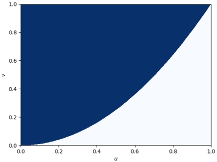
Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$



Are X and Y independent? Explain your answer:

FA12 Problem 6: Part (a)

Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$

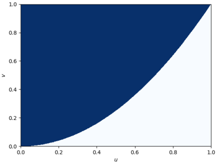


Are X and Y independent? Explain your answer:

X and Y are **not independent** since their support does not form a product set. We can prove this with $(a,b) = (1,1)$ and $(c,d) = (0,0)$ that lie in the support of (X,Y). These points fail the swap property since $(a,d) = (1,0) \notin \text{supp}(X,Y)$, thus X and Y are dependent.

FA12 Problem 6: Part (b)

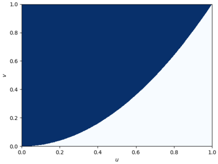
Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$



Determine $f_X(u)$:

FA12 Problem 6: Part (b)

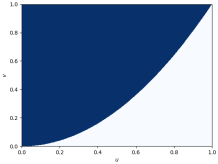
Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$



Determine $f_X(u)$:

$$f_X(u) = \int_{u^2}^{1} f_{X,Y}(u,v)dv$$

Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$

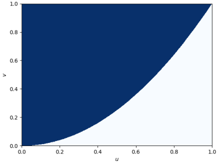


Determine $f_X(u)$:

$$\begin{aligned} f_X(u) &= \int_{u^2}^{1} f_{X,Y}(u,v)dv \\ &= \int_{u^2}^{1} \frac{3}{2}dv \\ &= \frac{3}{2}(1-u^2),\ 0 < u < 1 \end{aligned}$$

FA12 Problem 6: Part (c)

Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$



For what values of u is $f_{Y|X}(v|u)$ well defined?

FA12 Problem 6: Part (c)

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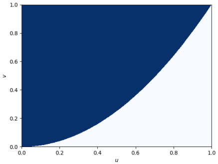
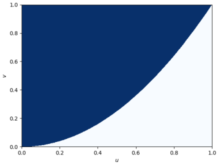


For what values of u is $f_{Y|X}(v|u)$ well defined?

Since $f_{Y|X}(v|u) = \frac{f_{X,Y}}{f_X}$, we need $f_X > 0$ for the conditional pdf to be well defined. From part (a), we see this means $u \in (0,1)$.

FA12 Problem 6: Part (d)

Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$



Determine $f_{Y|X}(v|u)$:

FA12 Problem 6: Part (d)

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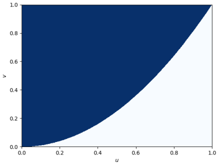

Determine $f_{Y|X}(v|u)$:

$$f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)}$$

FA12 Problem 6: Part (d)

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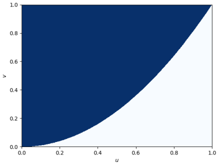
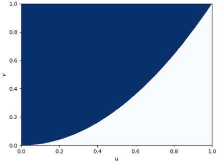


Determine $f_{Y|X}(v|u)$:

$$\begin{aligned} f_{Y|X}(v|u) &= \frac{f_{X,Y}(u,v)}{f_X(u)} \\ &= \frac{\frac{3}{2}}{\frac{3}{2}(1-u^2)} \\ &= \frac{1}{1-u^2},\ \ 0 < u < 1 \end{aligned}$$

FA12 Problem 6: Part (e)

Given $f_{X,Y}(u,v) = \begin{cases} \frac{3}{2}, & u > 0,\ u^2 < v < 1 \\ 0, & \text{else} \end{cases}$



Determine $P(Y > X)$

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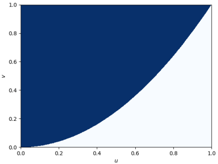

Determine $P(Y > X)$

We could take double integral or since $f_{X,Y}$ is uniform over supp(X, Y), we can take ratio of the area $Y > X$ (triangle) to the area of support.

FA12 Problem 6: Part (e)

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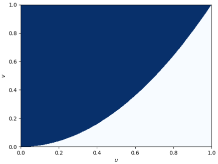

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$$\text{Area of support} \times \text{Weight of PDF} = 1$$
$$\text{Area of support} = \frac{2}{3}$$

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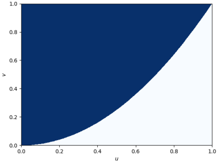

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$$\text{Area of } Y > X = \frac{1}{2}(1)(1) = \frac{1}{2}$$

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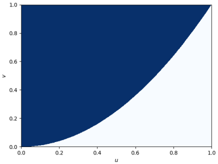

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$$P(Y > X) = \frac{1/2}{2/3} = \frac{3}{4}$$

Thanks everyone!
Stay safe, stay sane!
Good luck studying!