HKN ECE 310 Exam 1 Review

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Outline

- LSIC Systems
- BIBO Stability
- Impulse Response and Convolution
- Z-transform

LSIC Systems

For our system T,

- Linearity: must satisfy *homogeneity* and *additivity*.
  - Homogeneity: $T(ax) = aT(x)$
  - Additivity: $T(x+z) = T(x) + T(z)$
  - Can be summarized by *superposition*: $T(ax+bz) = aT(x) + bT(z)$
- Shift Invariance: shifting the input shifts the output by the same amount.
  - If $T(x[n]) = y[n]$, then $T(x[n-n_0]) = y[n-n_0]$.
  - $T(x[n-n_0])$ means we shift our input by n_0: we replace every n inside our input arguments with $n-n_0$.
  - $y[n-n_0]$ means we shift our output by n_0: we replace every *function* of n in our output with $f(n)-n_0$.
- Causality: output cannot depend on future input values.

BIBO Stability

Three ways to check for BIBO stability:

1. Absolute summability of the **impulse response**:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

2. System definition:
   - Given $|x[n]| < \alpha$, if $|T(x[n])| = |y[n]| < \beta < \infty$, T is BIBO stable
   - In other words, a bounded input yields a bounded output.
   - For example: $y[n] = x^5[n] + 3$ vs. $y[n] = x[n] * u[n]$
3. Pole-zero plot (more on this soon)

Convolution

$$x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$

The following statements are equivalent:

- Our system T is LSI.
- Our system T can be described by its impulse response $h[n]$.
- Our system response to an input signal is a convolution.

Also,

- If x is of length L and h is of length M, y must be of length $L + M - 1$.

Impulse Response

Let $x[n]$ be the input to an LSI system with impulse response $h[n]$. Then the system output $y[n]$ is given by:

$$y[n] = x[n] * h[n]$$

By the identity property of convolution, we can find $h[n]$ by passing a Kronecker delta $\delta[n]$ to our system:

$$h[n] = \delta[n] * h[n]$$

For example:

$$y[n] = 2x[n] - 3x[n-1] + x[n-2]$$
$$h[n] = 2\delta[n] - 3\delta[n-1] + \delta[n-2]$$

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- Typically perform inverse z-transform by inspection or by partial fraction decomposition
- Important properties:
  - Multiplication by n: $nx[n] \leftrightarrow -z\frac{dX(z)}{dz}$
  - Delay property: $y[n-k]u[n-k] \leftrightarrow z^{-k}Y(z)$
- Make sure to note the Region of Convergence (ROC) for your transforms, a Z-transform is not unique without one!
- Convolution theorem: $y[n] = x[n] * h[n] \overset{\mathcal{Z}}{\leftrightarrow} X(z)H(z) = Y(z)$

BIBO Stability Revisited

- An LSI system is BIBO stable if its ROC contains the unit circle.
- For causal systems, the ROC is anything *greater than* the outermost pole: $|z| > |p_{\max}|$
- For a non-causal system, the ROC is anything *less than* the innermost pole: $|z| < |p_{\min}|$
- If we sum multiple systems the ROC is the *intersection* of each system's ROC.

BIBO Stability Revisited

But what if the ROC is $|z| > 1$ or $|z| < 1$?

- This system is *marginally stable*... though sometimes we just call the system unstable anyway.
- For unstable systems, you are commonly asked to find a bounded input that yields an unbounded output. Few ways to do this:
  - Pick an input that excites the poles of a marginally stable system. This means we pick an input with frequency equal to the angle of the pole. For example, pole at $z = j$ is excited by $e^{j\frac{\pi}{2}n}$ or $\cos\left(\frac{\pi}{2}n\right)$
  - If the system's impulse response $h[n]$ is not absolutely summable, $u[n]$ works.
  - If the system's impulse response is unbounded, $\delta[n]$ works.

LSIC Examples

For the following systems, determine whether is is linear, shift-invariant and causal.

1. $y[n] = |x[n]|$
2. $y[n] = nx[n]$
3. $y[n] = x[n-1] + x[|n|]$
4. $\log(x[n])$

LSIC Examples: Part 1

$y[n] = |x[n]|$ is non-linear, shift-invariant, and causal.

- Linearity:

$$T(ax) \stackrel{?}{=} aT(x)$$
$$T(ax) = |ax[n]| \neq a|x[n]| = aT(x)$$
$$\implies \text{Non-linear}$$

- Shift-invariance:

$$T(x[n-n_0]) \stackrel{?}{=} y[n-n_0]$$
$$T(x[n-n_0]) = |x[(n-n_0)]| = |x[(n)-n_0]| = y[n-n_0]$$
$$\implies \text{Shift-invariant}$$

- Causality: our output time sample is always equal to our input time sample; therefore, this system is causal.

LSIC Examples: Part 2

$y[n] = nx[n]$ is linear, shift-varying, and causal.

- Linearity:

$$T(ax + bz) \stackrel{?}{=} aT(x) + bT(z)$$
$$T(ax + bz) = n(ax[n] + bz[n]) = anx[n] + bnz[n] = aT(x) + bT(z)$$
$$\implies \text{Linear}$$

- Shift-invariance:

$$T(x[n - n_0]) \stackrel{?}{=} y[n - n_0]$$
$$T(x[n - n_0]) = nx[(n - n_0)] \neq (n - n_0)x[(n) - n_0] = y[n - n_0]$$
$$\implies \text{Shift-varying}$$

- Causality: our output time sample is always equal to our input time sample; therefore, this system is causal. Don't for the n out front does not affect causality since it is not part of the input signal's argument!

LSIC Examples: Part 3

$y[n] = x[n-1] + x[|n|]$ is linear, shift-varying, and non-causal.

- Linearity:

$$T(ax+bz) \stackrel{?}{=} aT(x) + bT(z)$$
$$T(ax+bz) = (ax[n-1] + bz[n-1]) + (ax[|n|] + bz[|n|])$$
$$= ax[n-1] + ax[|n|] + bz[n-1] + bz[|n|] = aT(x) + bT(z)$$
$$\implies \text{Linear}$$

- Shift-invariance:

$$T(x[n-n_0]) \stackrel{?}{=} y[n-n_0]$$
$$T(x[n-n_0]) = x[(n-n_0)-1] + x[|n-n_0|]$$
$$\neq x[(n-1)-n_0] + x[(|n|)-n_0] = y[n-n_0]$$
$$\implies \text{Shift-varying}$$

- Non-causal: for example, $y[-3]$ relies on $x[3]$ in the future.

LSIC Examples: Part 4

$y[n] = \log(x[n])$ is non-linear, shift-invariant, and causal.

- Linearity:

$$T(ax) \stackrel{?}{=} aT(x)$$
$$T(ax) = \log(ax[n]) \neq a\log(x[n]) = aT(x)$$
$$\implies \quad \text{Non-linear}$$

- Shift-invariance:

$$T(x[n-n_0]) \stackrel{?}{=} y[n-n_0]$$
$$T(x[n-n_0]) = \log(x[(n-n_0)]) = \log(x[(n)-n_0]) = y[n-n_0]$$
$$\implies \quad \text{Shift-invariant}$$

- Causality: our output time sample is always equal to our input time sample; therefore, this system is causal.

BIBO Stability Example

For each of the following systems defined either by an input-output relationship or impulse response, determine whether the system is BIBO stable or not:

1. $h[n] = \delta[n]$
2. $h[n] = \left(-\frac{1}{3}\right)^n u[n]$
3. $y[n] = x^2[n] + 1$
4. $y[n] = (x[n])^a + b,\ 0 < a, b < c < \infty$

BIBO Stability Example

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3. $y[n] = x^2[n] + 1$: If $x[n] < c \implies x^2[n] + 1 < c^2 + 1 < \infty$; therefore, BIBO stable.

BIBO Stability Example

1. $h[n] = \delta[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
2. $h[n] = \left(-\frac{1}{3}\right)^n u[n]$: $h[n]$ is absolutely summable; therefore, BIBO stable.
3. $y[n] = x^2[n] + 1$: If $x[n] < c \implies x^2[n] + 1 < c^2 + 1 < \infty$; therefore, BIBO stable.
4. $y[n] = (x[n])^a + b,\ 0 < a, b < c < \infty$:
   $x[n] < c \implies (x[n])^a < c^a$. Moreover, adding $b < c < \infty$ will keep our output bounded below $c^a + b < \infty$; therefore, BIBO stable.

Impulse Response and Convolution Examples

Given $x[n] = \begin{bmatrix} 1 & 2 & 3 & 2 & 9 & 8 & 9 \end{bmatrix}$ and $h[n] = \begin{bmatrix} \mathbf{-1} & 0 & 1 \end{bmatrix}$, compute $y[n] = x[n] * h[n]$. Bonus: what does this filter do?

Suppose we have a digital filter $h[n]$ with an unknown impulse response. We do know the system output to the following two input signals. Determine the impulse response in terms of the two system outputs.

- $x_1[n] = \begin{bmatrix} 2 & 4 & 2 & 4 \end{bmatrix} \rightarrow y_1[n]$
- $x_2[n] = \begin{bmatrix} 0 & 2 & 1 & 2 \end{bmatrix} \rightarrow y_2[n]$

Impulse Response and Convolution Examples

Part 1: $x[n] = \begin{bmatrix} 1 & 2 & 3 & 2 & 9 & 8 & 9 \end{bmatrix}$ and $h[n] = \begin{bmatrix} -\mathbf{1} & 0 & 1 \end{bmatrix}$

$$y[n] = x[n] * h[n] = \begin{bmatrix} -1 & -2 & -2 & 0 & -6 & -6 & 0 & 8 & 9 \end{bmatrix}$$

Impulse Response and Convolution Examples

Part 1: $x[n] = \begin{bmatrix} 1 & 2 & 3 & 2 & 9 & 8 & 9 \end{bmatrix}$ and $h[n] = \begin{bmatrix} -\mathbf{1} & 0 & 1 \end{bmatrix}$

$$y[n] = x[n] * h[n] = \begin{bmatrix} -1 & -2 & -2 & 0 & -6 & -6 & 0 & 8 & 9 \end{bmatrix}$$

Part 2:

- $x_1[n] = \begin{bmatrix} 2 & 4 & 2 & 4 \end{bmatrix} \rightarrow y_1[n]$
- $x_2[n] = \begin{bmatrix} 0 & 2 & 1 & 2 \end{bmatrix} \rightarrow y_2[n]$

$$\begin{aligned} h[n] &= h[n] * \delta[n] \\ &= h[n] * (\frac{1}{2}x_1[n] - x_2[n]) \\ &= h[n] * \frac{1}{2}x_1[n] - h[n] * x_2[n] \\ &= \frac{1}{2}y_1[n] - y_2[n]. \end{aligned}$$

Combinations of Systems

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],\ \ h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

1. Suppose we pass input $x[n]$ to system $h[n]$ is the connection of $h_1[n]$ and $h_2[n]$ in series. Write the output $y[n]$ interms of $x[n], h_1[n]$ and $h_2[n]$.
2. Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of $h[n]$?
3. Suppose the two systems are now connected in parallel to form $h[n]$. Now what is the resulting transfer function and impulse response of $h[n]$?

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],\ \ h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose we pass input $x[n]$ to system $h[n]$ is the connection of $h_1[n]$ and $h_2[n]$ in series. Write the output $y[n]$ interms of $x[n], h_1[n]$ and $h_2[n]$.

$$y[n] = x[n] * h_1[n] * h_2[n]$$

Combinations of Systems: Part 2

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],\ h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of $h[n]$?

$$h[n] = h_1[n] * h_2[n]$$
$$\downarrow \mathcal{Z}$$
$$H(z) = H_1(z)H_2(z)$$
$$H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$
$$H_2(z) = \frac{\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}$$

Notice that $h_2[n] = \left(\frac{1}{3}\right)^n u[n-1] = \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} u[n-1]$

Combinations of Systems: Part 2

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n], \;\; h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose the two systems are still connected in series. What is the resulting transfer function and impulse response of $h[n]$?

$$\begin{aligned} H(z) &= \frac{\frac{1}{3}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} \\ &= \frac{K_1}{1-\frac{1}{2}z^{-1}} + \frac{K_2}{1-\frac{1}{3}z^{-1}} \\ &\overset{\text{PFD}}{\rightarrow} K_1 = 2, \;\; K_2 = -2 \end{aligned}$$

$$\text{Inspection/Tables} \\ \downarrow$$

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[n]$$

Combinations of Systems: Part 3

Given the following two LSI systems:

$$h_1[n] = \left(\frac{1}{2}\right)^n u[n],\ h_2[n] = \left(\frac{1}{3}\right)^n u[n-1]$$

- Suppose the two systems are now connected in parallel to form $h[n]$. Now what is the resulting transfer function and impulse response of $h[n]$?

$$\begin{aligned} h[n] &= h_1[n] + h_2[n] \\ &= \left(\frac{1}{2}\right)^n u[n] + \left(\frac{1}{3}\right)^n u[n-1] \\ H(z) &= H_1(z) + H_2(z) \\ &= \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}} \end{aligned}$$

Marginal Stability Example

Suppose we have a system response given by $H(z) = \frac{1}{1+z^{-3}}$. Which of the following bounded inputs would cause this system to have an unbounded output? There may be more than one!

1. $\cos\left(\frac{2\pi}{3}n\right) u[n]$
2. $\cos\left(\frac{\pi}{3}n\right) u[n]$
3. $u[n]$
4. $e^{-j\frac{\pi}{3}n}u[n]$
5. $(-1)^n u[n]$

Marginal Stability Example

$$H(z) = \frac{1}{1 + z^{-3}}$$

First, must find the poles of $H(z)$:

$$1 + z^{-3} = 0$$

$$z^{-3} = -1 = e^{-j\pi} = e^{-j(\pi + 2\pi k)}, \ \forall k \in \mathbb{Z}$$

$$z = e^{j\left(\frac{\pi}{3} + \frac{2\pi}{3}k\right)}$$

$$z_1 = e^{j\frac{\pi}{3}}, \ z_2 = e^{j\pi}, \ z_3 = e^{j\frac{5\pi}{3}}$$

Now let's look at our siginals...

Marginal Stability Example

$$H(z) = \frac{1}{1 + z^{-3}}$$

$$z_1 = e^{j\frac{\pi}{3}},\ z_2 = e^{j\pi},\ z_3 = e^{j\frac{5\pi}{3}}$$

1. $\cos\left(\frac{2\pi}{3}n\right) u[n]$
2. $\cos\left(\frac{\pi}{3}n\right) u[n]$ ✓
3. $u[n]$
4. $e^{-j\frac{\pi}{3}n}u[n]$ ✓
5. $(-1)^n u[n]$ ✓

Thank you

Good luck studying and good luck on your exam!